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Differential Equations Associated to a Representation of a Lie algebra from the Viewpoint of Nilpotent Analysis

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1 Introduction

If we generalize the notion of a manifold to that of a filtered manifold, the usual rôle of tangent space is played by the nilpotent graded Lie algebra which is defined at each point of the filtered manifold as its first order approximation. On the basis of this nilpotent approximation we have been studying various structures and objects on filtered manifolds to develop Nilpotent Geometry and Analysis.

In this paper we present a simple principle to associate systems of differential equations to a representation of a Lie algebra in the framework of nilpotent analysis.

2 Transitive graded Lie algebras, Representations and cohomology groups

Let $\mathfrak{g} = \bigoplus_{p\in\mathbf{Z}} \mathfrak{g}_p$ be a transitive graded Lie algebra, that is, a Lie algebra satisfying:

i) $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$

ii) $\dim \mathfrak{g}_- < \infty$, where $\mathfrak{g}_- = \bigoplus_{p<0} \mathfrak{g}_p$, the negative part of $\mathfrak{g}$

iii) (Transitivity) For $i \geq 0, x_i \in \mathfrak{g}_i$, if $[x_i, \mathfrak{g}_-] = 0$, then $x_i = 0$.

Let $V = \bigoplus_{q\in\mathbf{Z}} V_q$ be a graded vector space satisfying:

i) $\dim V_q < \infty$.

ii) There exists q_I such that $V_q = 0$ for $q \leq q_I$.

Let $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on V such that

(Λ1) $\lambda(\mathfrak{g}_p)V_q \subset V_{p+q}$.

(Λ2) There exists q_0 such that if $\lambda(\mathfrak{g}_-)x_q = 0$ for $q > q_0$ then $x_q = 0$.

We then consider the cohomology group $H(\mathfrak{g}_-, V) = \bigoplus_{p,r\in\mathbb{Z}} H^p_r(\mathfrak{g}_-, V)$ of the representation of $\mathfrak{g}_-$ on V, namely the cohomology group of the cochain complex:

$$\xrightarrow{\partial} \mathrm{Hom}(\wedge^{p-1}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p}\mathfrak{g}_-, V)_r \xrightarrow{\partial} \mathrm{Hom}(\wedge^{p+1}\mathfrak{g}_-, V)_r \xrightarrow{\partial}$$

where $\mathrm{Hom}(\wedge^p\mathfrak{g}_-, V)_r$ is the set of all homogeneous p-cochain ω of degree r, that is, $\omega(\mathfrak{g}_{a_1}\wedge\cdots\wedge\mathfrak{g}_{a_p}) \subset V_{a_1+\cdots+a_p+r}$ for any $a_1,\cdots,a_p<0$, and the coboundary operator ∂ is defined by

$$\begin{aligned}\partial\omega(X_1,\ldots,X_{p+1}) &= \sum(-1)^{i-1}\lambda(X_i)\omega(X_1,\ldots,\overset{\vee}{X_i},\ldots,X_{p+1})\\ &+ \sum(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\overset{\vee}{X_i},..,\overset{\vee}{X_j},\ldots,X_{p+1})\end{aligned}$$

for $\omega\in\mathrm{Hom}(\wedge^p\mathfrak{g}_-, V)_r$ and $X_1,\ldots,X_{p+1}\in\mathfrak{g}_-$.

Note that the condition (Λ2) above is equivalent to saying that

($\Lambda 2'$) $H^0_r(\mathfrak{g}_-, V) = 0$ for $r > q_0$.

This condition guarantees the finite dimensionality of the cohomology group; that is, there exists k_0 such that $H^p_r(\mathfrak{g}_-, V) = 0$ for $r \geq k_0$. (See [6]).

Now what we assert in this paper may be roughly stated as follows:

Principle *The first cohomology group $H^1(\mathfrak{g}_-, V) = \bigoplus H^1_r(\mathfrak{g}_-, V)$ represents a system of differential equations and $V = \bigoplus V_q$ represents its solution space.*

If the gradation of $\mathfrak{g}_-$ is trivial, that is, $\mathfrak{g}_- = \mathfrak{g}_{-1}$, then the cohomology group $H^p_r(\mathfrak{g}_-, V)$ is just the Spencer cohomology group, and in this case the above principle may be naturally accepted for those who are familiar to the formal theory of differential equations à la Spencer ([3], [12]) and there are related works ([11], [14], [1]).

We shall see that it is in the framework of nilpotent analysis that the principle above, in its general form, is properly and well settled. It then enables one to produce plenty of examples of systems of differential equations related to various geometric structures on filtered manifolds.

To formulate precisely the statement above we need some basic notions in nilpotent geometry and analysis, in particular, those of filtered manifolds, geometric structures on filtered manifolds, weighted jet bundles, and differential equations on filtered manifolds.

3 Filtered manifolds and geometric structures

A filtered manifold is a differential manifold M endowed with a filtration $\{\mathfrak{f}^p\}_{p\in\mathbb{Z}}$ consisting of subbundles $\mathfrak{f}^p$ of the tangent bundle TM such that

i) $\mathfrak{f}^p \supset \mathfrak{f}^{p+1}$,

ii) $\mathfrak{f}^0TM = 0$, $\quad\bigcup_{p\in\mathbb{Z}}\mathfrak{f}^p = TM$,

iii) $[\underline{\mathfrak{f}}^p, \underline{\mathfrak{f}}^q] \subset \underline{\mathfrak{f}}^{p+q}$ for all $p, q \in \mathbb{Z}$,

where $\underline{\mathfrak{f}}^p$ denotes the sheaf of the germs of sections of $\mathfrak{f}^p$.

There is associated to each point x of a filtered manifold $(M, \mathfrak{f})$ a graded object

$$gr\mathfrak{f}_x = \bigoplus_{p \in \mathbf{Z}} gr_p \mathfrak{f}_x, \quad \text{with} \quad gr_p\mathfrak{f}_x = \mathfrak{f}_x^p / \mathfrak{f}_x^{p+1},$$

which is not only a graded vector space but also has a natural Lie bracket induced from that of vector fields and proves to be a nilpotent graded Lie algebra.

A filtered manifold $(M, \mathfrak{f})$ is said to be of type $\mathfrak{g}_-$ if $gr\mathfrak{f}_x$ is isomorphic to a graded Lie algebra $\mathfrak{g}_-$ for all $x \in M$.

Let $(M, \mathfrak{f})$ be a filtered manifold of type $\mathfrak{g}_-$. We define $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)_x$ for $x \in M$ to be the set of all graded Lie algebra isomorphism $z : \mathfrak{g}_- \to gr\mathfrak{f}_x$, and set $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-) = \bigcup_{x \in M} \mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)_x$. Then $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)$ is a principal fibre bundle over M with structure group $\mathrm{Aut}_0(\mathfrak{g}_-)$, the group of automorphisms of the graded Lie algebra $\mathfrak{g}_-$ and is called the reduced frame bundle of $(M, \mathfrak{f})$.

Let G_0 be a Lie subgroup of $\mathrm{Aut}_0(\mathfrak{g}_-)$. A reduction of $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)$ to G_0 is a principal subbundle of $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)$ with structure group G_0, and is a geometric structure of first order on $(M, \mathfrak{f})$ of type $\mathfrak{g}_-$, alternatively called G_0-structure on $(M, \mathfrak{f})$.

4 Weighted jet bundles and differential equations

Let $(M, \mathfrak{f})$ be a filtered manifold. We say that a local vector field X on $(M, \mathfrak{f})$ is of weighted order $\leq k$ and write w-ord$X \leq k$ if X is a section of $\mathfrak{f}^{-k}$. A differential operator P on $(M, \mathfrak{f})$ is said to be of weighted order $\leq k$ and written w-ord$P \leq k$ if $P = \sum X_1 \cdots X_r$ (locally) for local vector fields $X_1, \cdots, X_r$ and if $\sum$ w-ord$X_i \leq k$.

Now consider a filtered vector bundle $(E, \{E^p\}_{p \in \mathbf{Z}})$ over a filtered manifold $(M, \mathfrak{f})$ such that

i) E^p is a vector bundle over M of rank finite.

ii) $E = E^{\nu_I} \supset \cdots \supset E^p \supset E^{p+1} \supset \cdots \supset E^{\nu_T + 1} = 0$.

Let $\underline{E}$ denote the sheaf of local sections of E and $\underline{E}_a$ the stalk over $a \in M$. First we define a filtration $\{\mathfrak{f}^k \underline{E}_a\}$ of $\underline{E}_a$ by setting $\mathfrak{f}^k \underline{E}_a$ to be the subspace of $\underline{E}_a$ consisting of $s \in \underline{E}_a$ such that $(P\langle \alpha^i, s\rangle)(a) = 0$ for any differential operator P and any section α^i of the annihilating bundle $(E^{i+1})^\perp$ of E^{i+1} whenever

$$\text{w-ord}P + i < k.$$

We then define:

$$\mathfrak{J}^k E = \bigcup_{a \in M} \mathfrak{J}^k_a E, \quad \mathfrak{J}^k_a E = \underline{E}_a / \mathfrak{f}^{k+1} \underline{E}_a.$$

We denote by j^k and j^k_a the natural projections $\underline{E} \to \mathfrak{J}^k E$ and $\underline{E}_a \to \mathfrak{J}^k_a E$ respectively. It is easy to see that $\mathfrak{J}^k E$ is a vector bundle over M. There is a natural filtration of $\mathfrak{J}^k E$ defined by $\mathfrak{f}^\ell \mathfrak{J}^k E = 0$ for $\ell \geq k+1$ and by the following exact sequences for $\ell \leq k$:

$$0 \longrightarrow \mathfrak{f}^{\ell+1}\mathfrak{J}^k E \longrightarrow \mathfrak{J}^k E \xrightarrow{\pi_{k\ell}} \mathfrak{J}^\ell E \longrightarrow 0,$$

where $\pi_{k\ell}$ are the natural projections. The vector bundle $\mathfrak{J}^k E$ equipped with this filtration will be called the weighted jet bundle of order k of $(E, \mathfrak{f})$ over $(M, \mathfrak{f})$.

The subbundle $\mathfrak{f}^k \mathfrak{J}^k E$ is called the symbol of $\mathfrak{J}^k E$ and given explicitly by the following fundamental exact sequence of bundle mappings:

$$0 \longrightarrow \mathrm{Hom}(U(gr\mathfrak{f}), grE)_k \longrightarrow \mathfrak{J}^k E \longrightarrow \mathfrak{J}^{k-1} E \longrightarrow 0.$$

Here for $x \in M$, we denote by grE_x the associated graded vector space to $\{E^p_x\}$ and by $U(gr\mathfrak{f}_x)$ the universal enveloping algebra of $gr\mathfrak{f}_x$. Remarking that $U(gr\mathfrak{f}_x)$ is graded: $U(gr\mathfrak{f}_x) = \bigoplus U_\ell$, where U_ℓ denotes the set of all homogeneous elements of degree ℓ ($\deg \xi = \sum p_i$ if $\xi = A_1 \cdots A_m$ with $A_i \in gr_{p_i}\mathfrak{f}_x$), we denote by $\mathrm{Hom}(U(gr\mathfrak{f}_x), grE_x)_k$ the set of all linear mapping $f : U(gr\mathfrak{f}_x) \to grE_x$ of degree k, namely $f(U_l) \subset gr_{\ell+k}E_x$. Thus in the above sequence $\mathrm{Hom}(U(gr\mathfrak{f}), grE)_k$ denotes the vector bundle whose fibre at x is $\mathrm{Hom}(U(gr\mathfrak{f}_x), grE_x)_k$.

Now some elementary properties are in order:

(1) Since the map $\mathrm{j}^k_x : \underline{E}_x \to \underline{\mathfrak{J}^k E}_x$ preserves the filtration, that is $\mathrm{j}^k_x(\mathfrak{f}^{\ell+1}\underline{E}_x) \subset \mathfrak{f}^{\ell+1}\underline{\mathfrak{J}^k E}_x$ for $\ell \in \mathbb{Z}$, we have the bundle map:

$$\iota : \mathfrak{J}^\ell E \to \mathfrak{J}^\ell \mathfrak{J}^k E.$$

(2) If $\varphi : (E, \{E^p\}) \to (F, \{F^q\})$ is a bundle map of degree r, that is, $\varphi(E^p) \subset F^{p+r}$ for all p, then it induces the bundle map for all ℓ:

$$\mathrm{j}^\ell \varphi : \mathfrak{J}^\ell E \to \mathfrak{J}^{\ell+r} F.$$

Now let us consider differential equations on a filtered manifold, confining our discussion to the linear case for the sake of simplicity. It is not difficult to extend the following discussions to the non-linear case.

Let $(E, \{E^p\})$ and $(F, \{F^q\})$ be filtered vector bundles over a filtered manifold $(M, \mathfrak{f})$. A bundle map (of degree r)

$$\Phi : \mathfrak{J}^k E \to F$$

is a linear differential operator of weighted order k and the kernel of Φ, denoted by R, is a system of linear differential equations. A section s of E is a solution of R if $\Phi(\mathrm{j}^k s) = 0$.

Without loss of generality we may assume that Φ is of degree 0 and $E^{k+1} = F^{k+1} = 0$.

If $\Phi : \mathfrak{J}^k E \to F$ is a bundle map of degree 0, it induces bundle maps for $i \leq k$:

$$\Phi^i : \mathfrak{J}^i E \longrightarrow F/F^{i+1}.$$

It then induces the symbol map:

$$gr_i\Phi : \mathfrak{f}^i\mathfrak{J}^i E (= \mathrm{Hom}(U(gr\mathfrak{f}), grE)_i) \to \mathfrak{f}^i F^{(i)} (= gr_i F),$$

which we write:

$$gr\Phi : \mathrm{Hom}(U(gr), grE) \to grF.$$

We call Φ^i(or $R^i = Ker\Phi^i$) differential operator(or equation) associated to Φ (or R resp.), $gr\Phi$ the symbol map associated to Φ. We denote $Ker\Phi$ by $\mathfrak{s}(\Phi) = \bigoplus \mathfrak{s}_i(\Phi)$ and call it the symbol of Φ.

A bundle map $\Phi : \mathfrak{J}^k E \to F$ of degree 0 gives rise to the bundle maps for all ℓ:

$$p^\ell\Phi : \mathfrak{J}^\ell E \xrightarrow{\iota} \mathfrak{J}^\ell\mathfrak{J}^k E \xrightarrow{j^\ell\Phi} \mathfrak{J}^\ell F,$$

simply denoted by $p(\Phi) : \mathfrak{J}E \to \mathfrak{J}F$ and called the prolongation of Φ. Note that a section of E is a solution of Φ if and only if it is a solution of $p\Phi^\ell$ for an $\ell \geq k$.

Note that $\mathrm{Hom}(U(gr\mathfrak{f}), grE)$ is a right $U(gr\mathfrak{f})$-module by

$$< \alpha\xi, \eta > = < \alpha, \xi\eta > \tag{1}$$

and left $U(gr\mathfrak{f})$-module by

$$< \eta, \xi\alpha > = < \eta\xi, \alpha > \tag{2}$$

for $\alpha \in \mathrm{Hom}(U(gr\mathfrak{f}, grE)$ and $\xi, \eta \in U(gr\mathfrak{f})$. We then have:

Proposition 1 *If $\Phi : \mathfrak{J}^k E \to F$ is a bundle map of degree 0, then the symbol map of the prolongation:*

$$gr(p\Phi) : Hom(U(gr\mathfrak{f}), grE) \to Hom(U(gr\mathfrak{f}), grF)$$

is a right $U(gr\mathfrak{f})$-homomorphism. Hence the symbol $\mathfrak{s}(p\Phi) = \bigoplus \mathfrak{s}_\ell(p\Phi)$ is a right $U(gr\mathfrak{f})$-module.

This proposition is fundamental for the formal theory of differential equations on filtered manifolds (See [10]).

We say a system of differential equation Φ is of finite type if the symbol of its prolongation $\mathfrak{s}(p\Phi)$ is finite dimensional, that is, there exists a k_0 such that $\mathfrak{s}_\ell(p\Phi) = 0$ for $\ell > k_0$.

A system of finite type can be essentially reduced to a system of ODE.

For a general existence theorem of an analytic solution to a system of infinite type, see [9], [10].

5 Differential equations associated to a representation

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a transitive graded Lie algebra, $V = \bigoplus_{q\in\mathbb{Z}} V_q$ a graded vector space, and $\lambda : \mathfrak{g} \to \mathfrak{gl}(V)$ a representation of $\mathfrak{g}$ on V as in the preceding sections.

Let $U(\mathfrak{g}_-)$ or simply U denote the universal enveloping algebra of $\mathfrak{g}_-$. Note that the set of all left $U(\mathfrak{g}_-)$-homomorphisms of $U(\mathfrak{g}_-)$ to V, denoted by $\mathrm{Hom}_U(U(\mathfrak{g}_-), V)$, is a left $U(\mathfrak{g}_-)$-module. (If V' is a right U-module, then the set of all right $U(\mathfrak{g}_-)$-homomorphisms of $U(\mathfrak{g}_-)$ to V is a right $U(\mathfrak{g}_-)$-module and denoted by $\mathrm{Hom}(U(\mathfrak{g}_-), V)_U$.)

Now define a mapping

$$\Lambda : V \to \mathrm{Hom}_U(U(\mathfrak{g}_-), V)$$

by

$$< \xi, \Lambda(v) >= \xi v \quad \text{for } \xi \in U,\ v \in V,$$

which is clearly a left U-isomorphism.

We set

$$I^a U = \{\xi \in U : deg\, \xi \leq a\},$$

and we have the following commutative diagram for $s \geq r$:

$$\begin{array}{ccccc}
V_s & \xrightarrow{\Lambda} & \mathrm{Hom}_U(U(\mathfrak{g}_-), V)_s & \xrightarrow{\theta} & \mathrm{Hom}_U(I^{q_0 - s}U(\mathfrak{g}_-), V)_s \\
L_\xi \downarrow & & L_\xi \downarrow & & L_\xi \downarrow \\
V_r & \xrightarrow{\Lambda} & \mathrm{Hom}_U(U(\mathfrak{g}_-), V)_r & \xrightarrow{\theta} & \mathrm{Hom}_U(I^{q_0 - r}U(\mathfrak{g}_-), V)_r
\end{array}$$

where θ denotes the restriction map and L_ξ denotes the left multiplication by ξ. Now we set

$$W = \bigoplus_{q \leq q_0} V_q$$

Then we see

$$\mathrm{Hom}_U(I^{q_0 - r}U(\mathfrak{g}_-), V)_r = \mathrm{Hom}_U(I^{q_0 - r}U(\mathfrak{g}_-), W)_r \subset \mathrm{Hom}(U(\mathfrak{g}_-), W)_r$$

and

$$\mathrm{Hom}_U(I^{q_0 - r}U(\mathfrak{g}_-), V)_r = V_r \quad \text{for } r \leq q_0.$$

For $r > q_o$, by the condition ($\Lambda 2$), the restriction maps

$$V_r \to \mathrm{Hom}_U(I^{-1}U(\mathfrak{g}_-), V)_r \to \mathrm{Hom}_U(I^{q_0 - r}U(\mathfrak{g}_-), V)_r$$

are injective. We have also

$$\mathrm{Hom}_U(I^{-1}U(\mathfrak{g}_-), V)_r \cong Z\mathrm{Hom}(\mathfrak{g}_-, V)_r,$$

where the latter space denotes the set of cocycles, that is the kernel of ∂ : $\mathrm{Hom}(\mathfrak{g}_-, V)_r \to \mathrm{Hom}(\wedge^2\mathfrak{g}_-, V)_r$. Hence we have:
For $r \le q_0$

$$V_r \underset{\cong}{\longrightarrow} \mathrm{Hom}_U(U, V)_r \underset{\cong}{\longrightarrow} \mathrm{Hom}_U(I^{q_0-r}U, V)_r \hookrightarrow \mathrm{Hom}(U, W)_r$$

For $r > q_0$

$$\begin{array}{ccccccc} & & & & \mathrm{Hom}_U(I^{q_0-r}U, V)_r & \hookrightarrow & \mathrm{Hom}(U, W)_r \\ & & & & \cup & & \\ V_r & \underset{\cong}{\longrightarrow} & \mathrm{Hom}_U(U, V)_r & \hookrightarrow & \mathrm{Hom}_U(I^{-1}U, V)_r & & \\ & & & & \|\downarrow & & \\ 0 & \longrightarrow & V_r & \to & Z\mathrm{Hom}(\mathfrak{g}_-, V)_r & \longrightarrow & H^1_r(\mathfrak{g}_-, V) \end{array}$$

It being prepared, we define

$$\mathfrak{s} = \bigoplus \mathfrak{s}_r, \quad \text{with} \quad \mathfrak{s}_r \subset \mathrm{Hom}(U(\mathfrak{g}_-), W)_r$$

by the following conditions:
(0) For $r \le q_0$ $\quad \mathfrak{s}_r = V_r$.
(1) For $r > q_0$

$$\mathfrak{s}_r \subset \mathrm{Hom}_U(I^{-1}U(\mathfrak{g}_-), V)_r \subset \mathrm{Hom}(U(\mathfrak{g}_-), W)_r \tag{3}$$
$$0 \to \mathfrak{s}_r \to Z\mathrm{Hom}(g_-, V)_r \to H^1_r(\mathfrak{g}_-, V) \to 0 \quad \text{(exact).} \tag{4}$$

Then we have

$$\mathfrak{s} = V.$$

This means that (3) and (4) above may be regarded as defining equations of V_r ($r > q_0$).

Let G_0 be a Lie subgroup of $\mathrm{Aut}_0(\mathfrak{g}_-)$ with Lie algebra $\mathfrak{g}_0$ and assume that the representation of $\mathfrak{g}_0$ is integrated to a representation of G_0. Let $(M, \mathfrak{f})$ be a filtered manifold of type $\mathfrak{g}_-$ on which there is given a G_0-structure $P^{(0)} \to M \subset \mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)$.

In general, if X is a left G_0-module, then we can construct the associated vector bundle $(P^{(0)} \times X)/G_0$ on M, which we denote by $M * X$. Note that $M * \mathfrak{g}_-$ is nothing but $gr\mathfrak{f}$. Therefore all the preceding discussions on left $U(\mathfrak{g}_-)$ module V are translated to that on left $U(gr\mathfrak{f})$-module $M * V$. Hence we could define a class of systems of differential equations on M whose symbols are specified by V: The left $U(gr\mathfrak{f})$-module $M * V$ is embedded in $\mathrm{Hom}(U(gr\mathfrak{f}), M * V)$ as left $U(gr\mathfrak{f})$-module whose defining equations are given by $H^1(gr\mathfrak{f}, M * V) = M * H^1(\mathfrak{g}_-, V)$. However, according to our convention, the symbols of prolonged equations are right $U(gr\mathfrak{f})$-modules (Proposition 1). So we need to switch from left to right. In general, for a Lie algebra A we have an involutive anti-isomorphism γ of $U(A)$ determined by: $\gamma(1) = 1$, $\gamma(x) = -x$ for $x \in A$, and $\gamma(\xi\eta) = \gamma(\eta)\gamma(\xi)$ for $\xi, \eta \in U(A)$. If B is a left $U(A)$-module, then it can be converted to a right $U(A)$-module by $vx = \gamma(x)v$ for $x \in A$ and $v \in B$.

In this way we regard $M * V$ as a right $U(gr\mathfrak{f})$-module and let it be embedded into $\mathrm{Hom}(U(gr\mathfrak{f}), M * V)$ as right $U(gr\mathfrak{f})$-module whose defining equations are given by $H^1(gr\mathfrak{f}, (M * V)', \partial') = M * H^1(\mathfrak{g}_-, V', \partial')$, where the prime ' indicates that it is considered as right module. The coboundary operator ∂' is defined for right $\mathfrak{g}_-$ module V' by

$$\begin{aligned} \partial\omega(X_1, \ldots, X_{p+1}) &= \sum (-1)^i \omega(X_1, \ldots, \overset{\vee}{X_i}, \ldots, X_{p+1}) X_i \\ &+ (-1)^{i+j} \omega([X_i, X_j] X_1, \ldots, \overset{\vee}{X_i}, .. \overset{\vee}{X_j}, .., X_{p+1}) \end{aligned}$$

for $\omega \in \mathrm{Hom}(\wedge^p \mathfrak{g}_-, V')_r$ and $X_1, \ldots, X_{p+1} \in \mathfrak{g}_-$. Then we see

$$H(\mathfrak{g}_-, V, \partial) = H(\mathfrak{g}_-, V', \partial').$$

We are now in a position to define a class of systems of differential equations $\mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$. Let q_1 be the smallest integer such that $H^1_q(\mathfrak{g}_-, V) = 0$ for $q > q_1$.

Definition 1 *We say a system of differential equations $R \subset \mathcal{J}^{q_1}(M * W)$ is of symbol type $\bigoplus_{q \le q_1} V_q$ (or the symbol of R is defined by $H^1(\mathfrak{g}_-, V)$) and denote $R \in \mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$ if the symbol $\mathfrak{s}_q(R) = (M * V)'_q$ for $q \le q_1$*

Thus a representation of $\mathfrak{g}$ on V determines a class $R \in \mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$ of systems of differential equations on a filtered manifold $(M, \mathfrak{f})$ of type $\mathfrak{g}_-$ on which a G_0 -structure $P^{(0)}$ is given.

In other word, a system of differential equations $R \in \mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$ is characterized by the property that its symbol has the form determined by $(\mathfrak{g}_-, V)$.

It is therefore clear that for $R \in \mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$ the symbol of its prolongation $\mathfrak{s}(pR)$ is contained in $(M * V)'$, and if all the compatibility conditions are satisfied in the course of prolongation then $\mathfrak{s}(pR) = (M * V)'$.

In particular, if $\dim V < \infty$ then $R^{q_1} \in \mathcal{S}_{(\mathfrak{g}_-, V, M, \mathfrak{f}, P^{(0)})}$ is of finite type. Let q_T be the smallest integer such that $V_q = 0$ for $q > q_T$. Then $\mathfrak{s}_q(pR) = 0$ for $q > q_T$ and the prolonged equation $p^q R$ can be written in such a solved form that all the derivatives of weighted order q is expressed in terms of lower order derivatives. Thus the solution space of R is of finite dimension $\le \dim V$.

For a given system of differential equations Φ the symbol $\mathfrak{s}(p\Phi)$ of $p\Phi$ is determined from that of Φ purely algebraically. Therefore deciding whether a system is finite type or not is an algebraic problem, which however often involves awful computations.

The advantage of starting from a representation $(\mathfrak{g}_-.V)$ is to avoid the direct computation of prolongation and to reduce it to the computation of cohomology groups.

In the case where $\mathfrak{g}$ is simple the cohomology groups can be computed by Kostant's generalized Borel-Weil theory ([4]).

6 Differential equations on bi-Legendrian manifolds

As an example let us consider $\mathfrak{g} = \mathfrak{sl}(n+2, K)$ with $K = \mathbb{C}$ or $\mathbb{R}$, and define a gradation

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2$$

by the eigen space decomposition of adJ, where J is the matrix $(a_{ij})_{0 \leq i,j \leq n+1}$ such that $a_{00} = 1, a_{n+1 n+1} = -1$ and $a_{ij} = 0$ for the others. Thus the gradation is described by the following figure:

$$\left(\begin{array}{c|c|c} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \hline \mathfrak{g}^X_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \hline \mathfrak{g}_{-2} & \mathfrak{g}^Y_{-1} & \mathfrak{g}_0 \end{array}\right)$$

Note that the negative part $\mathfrak{g}_-(= \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1})$ is isomorphic to the Heisenberg Lie algebra of dimension $2n+1$, and we have a direct sum decomposition

$$\mathfrak{g}_{-1} = \mathfrak{g}^X_{-1} \oplus \mathfrak{g}^Y_{-1}$$

as in the figure above into $\mathfrak{g}_0$-irreducible subspaces. We have

$$[\mathfrak{g}^X_{-1}, \mathfrak{g}^X_{-1}] = [\mathfrak{g}^Y_{-1}, \mathfrak{g}^Y_{-1}] = 0,$$

Hence $\mathfrak{g}^X_{-1}$ and $\mathfrak{g}^X_{-1}$ are Legendrian subspaces of $\mathfrak{g}_{-1}$. We denote by $Der_0(\mathfrak{g}_-)$ the Lie algebra of all derivations of degree 0. Then

$$\mathfrak{g}_0 \cong \{\alpha \in Der_0(\mathfrak{g}_-) \mid \alpha(\mathfrak{g}^X_{-1}) \subset \mathfrak{g}^X_{-1},\ \alpha(\mathfrak{g}^Y_{-1}) \subset \mathfrak{g}^Y_{-1}\}$$

We know that the prolongation of $\mathfrak{g}_-$ is the infinite dimensional contact Lie algebra, and the prolongation of $\mathfrak{g}_- \oplus \mathfrak{g}_0$ is, as easily verified, isomorphic to $\mathfrak{g}$.

Now let $V = K^{n+2}$ and consider the standard representation of $\mathfrak{g}$ on V. If we denote by $\{e_0, e_1, \cdots, e_{n+1}\}$ the standard basis of V and set

$$V_1 =< e_0 >, V_0 =< e_1, \cdots, e_n >, V_{-1} =< e_{n+1} >$$

Then we have $V = \bigoplus V_q$ and satisfies $\lambda(\mathfrak{g}_p)V_q \subset V_{p+q}$.

We then consider the cohomology group $H^p_r(\mathfrak{g}_-, V)$ of the representation of $\mathfrak{g}_-$ on V. By a simple computation we have:

Proposition 2 *The representation of $\mathfrak{g}_-$ on V being as above, we have*

$$H^1(\mathfrak{g}_-, V) = H^1_0(\mathfrak{g}_-, V) \oplus H^1_1(\mathfrak{g}_-, V)$$

and

$$H^1_0(\mathfrak{g}_-, V) \cong Hom(\mathfrak{g}^X_{-1}, V_{-1}),\ H^1_1(\mathfrak{g}_-, V) \cong Hom(S^2\mathfrak{g}^Y_{-1}, V_{-1}),$$

where $S^2\mathfrak{g}^Y_{-1}$ denotes the two-times symmetric tensor product of $\mathfrak{g}^Y_{-1}$.

Let $G = SL(n+2, K)$ and for $k \geq 0$ let F^kG be the largest subgroup of G whose Lie algebra is $F^k\mathfrak{g}$, where we set $F^k\mathfrak{g} = \bigoplus_{p \geq k} \mathfrak{g}_p$. We denote by Q the homogeneous space G/F^0G. It is a model space of the filtered manifolds of type $\mathfrak{g}_-$ having geometric structures of type F^0G/F^1G. There is a unique left invariant tangential filtration $\{\mathfrak{f}^p\}$ on Q which coincides with $\{F^p\mathfrak{g}/F^0\mathfrak{g}\}$ at the origin. Clearly it is of type $\mathfrak{g}_-$, and therefore $\mathfrak{f}^{-1}$ is a contact structure. Moreover, the decomposition $\mathfrak{g}_{-1} = \mathfrak{g}_{-1}^X \oplus \mathfrak{g}_{-1}^Y$ defines the decomposition $\mathfrak{f}^{-1} = \mathfrak{f}_X^{-1} \oplus \mathfrak{f}_Y^{-1}$ into Legendrian subbundles. The principal bundle $G/F^1G \to Q$ defines a standard geometric structure on Q of type F^0G/F^1G.

In this case these structures can be seen more concretely. The homogeneous space Q is the flag manifold consisting of all pairs $q = (\eta_1, \eta_2)$ of subspaces of V with $\dim \eta_1 = 1$, $\dim \eta_2 = n+1$ and $\eta_1 \subset \eta_2$. The mappings which send q to η_1 and η_2 define projections $\pi_1 : Q \to P(V)$ and $\pi_2 : Q \to P(V)^*$ respectively and

$$Q \cong \{([v], [\alpha]) \in P(V) \times P(V^*); < v, \alpha >= 0\}.$$

Moreover Q is canonically identified with $PT^*P(V)$, the projective cotangent bundle of the projective space $P(V)$, which has a canonical contact structure D given by

$$D = Ker(\pi_2)_* \oplus Ker(\pi_1)_*$$

We see easily that $Ker(\pi_2)_* = \mathfrak{f}_X^{-1}$, $Ker(\pi_1)_* = \mathfrak{f}_Y^{-1}$. Therefore the contact structure D coincides with $\mathfrak{f}^{-1}$.

The exponential mapping $\mathfrak{g}_- \to G$ composed with the projection on to Q gives a local diffeomorphism from $\mathfrak{g}_-$ into Q, which defines local coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n, z)$ of the point of Q corresponding to

$$\left(\begin{array}{c|ccc|c} 0 & & 0 & & 0 \\ \hline x^1 & & & & \\ \vdots & & 0 & & 0 \\ x^n & & & & \\ \hline z & y^1 & \cdots & y^n & 0 \end{array}\right)$$

Then the contact structure D is defined by :

$$\omega = dz + \frac{1}{2}\sum(-y^i dx^i + x^i dy^i) = 0.$$

The Legendre subbundles $\mathfrak{f}_X^{-1}$ and $\mathfrak{f}_Y^{-1}$ are spanned respectively by

$$\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} + \frac{1}{2}y^i\frac{\partial}{\partial z}\} \text{ and } \{\frac{\delta}{\delta y^i} = \frac{\partial}{\partial y^i} - \frac{1}{2}x^i\frac{\partial}{\partial z}\}$$

Now let us see what is the differential equations that the representation of $\mathfrak{g}$ on V determines on the homogeneous space Q. We note that the representation of $\mathfrak{g}_0$ on V is integrated to a representation of F^0G/F^1G on V. Since we have F^0G/F^1G -principal bundle G/F^1G over Q, the F^0G/F^1G -module $V = \bigoplus V_i$

defines the associated vector bundle $E_V = \bigoplus E_{V_i}$. Note that

$$\begin{aligned} V_0 &\cong \operatorname{Hom}(\mathfrak{g}_{-1}^Y, V_{-1}) \subset \operatorname{Hom}(U(\mathfrak{g}_-), v_{-1})_0 \\ V_1 &\cong \operatorname{Hom}(\mathfrak{g}_{-1}^X, V_0) \cong \operatorname{Hom}(\mathfrak{g}_{-1}^X \otimes \mathfrak{g}_{-1}^Y, V_{-1}) \subset \operatorname{Hom}(U(\mathfrak{g}_-), v_{-1})_1. \end{aligned}$$

Differential equations on Q defined by $H^1(\mathfrak{g}_-, V)$ are differential equations for a section of $E_{V_{-1}}$ written in terms of local coordinates in the following form:

$$\begin{cases} \dfrac{\delta}{\delta x^i} u &= f_i(x, y, z, u) \\ \dfrac{\delta^2}{\delta y^i \delta y^j} u &= f_{ij}(x, y, z, u, \frac{\delta u}{\delta x^i}, \frac{\delta u}{\delta y^i}), \end{cases}$$

where f_i, f_{ij} are arbitrary functions.

If $f_i, f_{i,j}$ both identically vanish, the solutions are given by

$$u = a + \sum b_i y^i + c(z - \frac{1}{2} \sum x^i y^i)$$

In our case of $\mathfrak{g} = \mathfrak{sl}(n+2)$ with the contact gradation $\mathfrak{g} = \bigoplus_{p=-2}^{2} \mathfrak{g}_p$, a filtered manifold $(M, \mathfrak{f})$ of type $\mathfrak{g}_-$ is nothing but a contact manifold, namely $\mathfrak{f}^{-1}$ is a contact distribution on M. Let $\mathcal{R}^{(0)}(M, \mathfrak{f}, \mathfrak{g}_-)$ be the reduced frame bundle of $(M, \mathfrak{f})$ of weighted order 1, that is, the fibre $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)_x$ on $x \in M$ is the set of all graded Lie algebra automorphisms $z : \mathfrak{g} \to gr\mathfrak{f}_x$ It is a principal fibre bundle on M with structure group $\mathrm{Aut}_0(\mathfrak{g}_-)$, the group of automorphisms of the graded Lie algebra $\mathfrak{g}_-$ (degree preserving). Note that F^0G/F^1G is a closed Lie subgroup of $\mathrm{Aut}_0(\mathfrak{g}_-)$. A principal subbundle $P^{(0)}$ of $\mathcal{R}^{(0)}(M, \mathfrak{f}; \mathfrak{g}_-)$ with structure group F^0G/F^1G is a first order geometric structure on $(M, \mathfrak{f})$ of type F^0G/F^1G, which turns out to be a bi-Legendrian structure on M in the following sense.

Definition 2 *A bi-Legendrian structure on a manifold M (or on a contact manifold (M, D)) is a triple (D, L_1, L_2) (resp. pair (L_1, L_2)) of subbundles of the tangent bundle TM of M such that*

$$D = L_1 \oplus L_2,$$

that D is a contact distribution and that L_1 and L_2 are Legendre subbundles of D. A bi-Legendrian (contact) manifold is a manifold equipped with a bi-Legendrian structure.

Remark 1 Let (M, D) be a contact manifold of dimension $2n+1$. Giving a subbundle E of D of rank n is equivalent to defining a Monge-Ampère equation on (M, D) which is decomposable in the sense of Machida-Morimoto (see [5]). Hence a bi-Legendrian structure (L_1, L_2) on a contact manifold (M, D) defines two Monge-Ampère equations L_1, L_2 on (M, D) which are decomposable and parabolic.

Remark 2 Since the prolongation of $\mathfrak{g}_- \oplus \mathfrak{g}_0$ is $\mathfrak{g}$ and simple, to each bi-Legendrian structure on a manifold M we can construct a Cartan connection modeled after $G \to G/F^0G$ ([13])

According to the prescription explained in the preceding section, we can define the class of systems of differential equations that the representation of $\mathfrak{g}$ on V determines on a bi-Legendrian manifold $(M, \mathfrak{f}, \mathfrak{f}_X^{-1}, \mathfrak{f}_Y^{-1})$.

It should be remarked that the unknown function of a system of differential equations belonging to this class is thus a section of $M * V_{-1}$ on M, which may be regarded as a contact vector field. In fact, we note that $V_{-1} * M$ can be identified with $gr_{-2}\mathfrak{f} = TM/D$ and the sections of TM/D can be identified with the infinitesimal contact transformations (contact vector fields) of (M, D).

Next let us consider the tensor representation of $\mathfrak{g} = \mathfrak{sl}(n+2, K)$ on the symmetric tensor product $W = S^2V = S^2K^{n+2}$. If we put $W_q = \bigoplus_{i+j=q} V_i \otimes V_j$, then we have $W = \bigoplus W_q$ and $\mathfrak{g}_p W_q \subset W_{p+q}$. By computation we have:

Proposition 3 *The representation of* $\mathfrak{g}_-$ *on* W *being as above, we have*

$$H^1(\mathfrak{g}_-, W) = H^1_{-1}(\mathfrak{g}_-, W) \oplus H^1_1(\mathfrak{g}_-, W)$$

and

$$H^1_{-1}(\mathfrak{g}_-, W) \cong Hom(\mathfrak{g}^X_{-1}, W_{-2}), \ \ H^1_1(\mathfrak{g}_-, W) \cong Hom(S^3\mathfrak{g}^Y_{-1}, W_{-2}),$$

where $S^3\mathfrak{g}^Y_{-1}$ *denotes the three-times symmetric tensor product of* $\mathfrak{g}^Y_{-1}$.

The systems of differential equations on the homogeneous space $Q = G/F^0G$ associated to the above representation have the following local expression:

$$\left\{ \begin{array}{lcl} \dfrac{\delta}{\delta x^i} u & = & f_i \\ \dfrac{\delta^3}{\delta y^i \delta y^j \delta y^k} u & = & f_{ijk}, \end{array} \right.$$

where f_i is an arbitrary functions of x, y, z, u, and f_{ijk} is an arbitrary function of x, y, z and the derivatives of u of which weighted orders are less than 3 .

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